

PSEUDO-RIEMANNIAN JACOBI-VIDEV MANIFOLDS

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ABSTRACT. We exhibit several families of Jacobi–Videv pseudo-Riemannian manifolds which are not Einstein. We also exhibit Jacobi–Videv algebraic curvature tensors where the Ricci operator defines an almost complex structure.

1. INTRODUCTION

Studying additional algebraic properties of the curvature tensor and relating these properties to the underlying geometry is an active field of investigation recently. Although Lorentzian geometry plays a central role in mathematical physics, the higher signature context is important as well. We refer to [1, 2] for a discussion of signature $(2, 2)$ Walker metrics; these are manifolds which admit a parallel totally isotropic 2-plane field. Dual and anti-self dual metrics are discussed by [3, 4] in the higher signature setting. Manifolds with a nilpotent Ricci operator of higher order appear naturally [5], and manifolds of signatures other than Riemannian or Lorentzian are important in Brane theory [6].

In this paper, we shall study when the Ricci operator and the Jacobi operator commute. Let ∇ be the Levi-Civita connection of a pseudo-Riemannian manifold (M, g) of signature (p, q) ; we shall primarily be concerned with the case $p \geq 1$ and $q \geq 1$. Let $\mathcal{R}(x, y) := \nabla_x \nabla_y - \nabla_y \nabla_x - \nabla_{[x, y]}$ be the Riemann curvature operator, let $\mathcal{J}(x) : y \rightarrow \mathcal{R}(y, x)x$ be the Jacobi operator, and let ρ be the Ricci operator. Following seminal work of Videv, we say that \mathcal{M} is *Jacobi–Videv* if $\mathcal{J}(x)\rho = \rho\mathcal{J}(x)$ for all x .

Clearly if \mathcal{M} is Einstein, i.e. if $\rho = \text{cid}$, then \mathcal{M} is Jacobi–Videv. If \mathcal{M} is indecomposable, the converse implication holds in the Riemannian setting; any indecomposable Riemannian Jacobi–Videv manifold is necessarily Einstein [7]; we also refer to related work [8]. This implication fails in the indefinite context. One has the following family of examples which are Jacobi–Videv and not Einstein. Manifolds in this family have been studied previously in different contexts, see for example [9, 10, 11, 12, 13]; we also refer to [14, 15]

Definition 1.1. Let $k \geq 1$, let $\ell \geq 1$, and $m = 2k + \ell$. Introduce coordinates

$$(x_1, \dots, x_k, y_1, \dots, y_\ell, \bar{x}_1, \dots, \bar{x}_k) \quad \text{on} \quad \mathbb{R}^m.$$

Let indices i, j range from 1 through k and index the collections $\{\partial_{x_1}, \dots, \partial_{x_k}\}$ and $\{\partial_{\bar{x}_1}, \dots, \partial_{\bar{x}_k}\}$. Let indices a, b range from 1 through ℓ and index the collection $\{\partial_{y_1}, \dots, \partial_{y_\ell}\}$. Let $S^2(\mathbb{R}^k)$ be the set of symmetric 2-tensors on \mathbb{R}^k and let ψ be a smooth map from \mathbb{R}^ℓ to $S^2(\mathbb{R}^k)$. Let $C_{ab} = C_{ba}$ define a non-degenerate inner product of signature (u, v) on \mathbb{R}^ℓ where $u + v = \ell$. We use ψ and C to define a pseudo-Riemannian manifold $\mathcal{M} = \mathcal{M}_{C, \psi} := (\mathbb{R}^{2k+\ell}, g_{C, \psi})$ where $g = g_{C, \psi}$ is the pseudo-Riemannian manifold of signature $(k + u, k + v)$ with non-zero components

$$g(\partial_{x_i}, \partial_{x_j}) := -2\psi_{ij}(\vec{y}), \quad g(\partial_{y_a}, \partial_{y_b}) = C_{ab}, \quad g(\partial_{x_i}, \partial_{\bar{x}_i}) = 1.$$

Theorem 1.2. *The manifold $\mathcal{M}_{C, \psi}$ of Definition 1.1 is Jacobi–Videv with nilpotent Ricci operator ρ . Furthermore, $\mathcal{M}_{C, \psi}$ is Einstein if and only if for any i, j with $1 \leq i, j \leq k$ we have $\sum_{ab} C^{ab} \partial_{y_a} \partial_{y_b} \psi_{ij} = 0$.*

We note that if ψ is a periodic function, then the metric g descends to define a metric on the torus $\mathbb{T}^{2k+\ell}$. Thus there are compact examples of Jacobi–Videv manifolds which are not Einstein.

One says that a pseudo-Riemannian manifold \mathcal{M} is *curvature homogeneous* if given any two points P and Q of M , there is an isometry $\phi_{P,Q}$ from $T_P M$ to $T_Q M$ so that $\phi_{P,Q}^* R_Q = R_P$.

Although a pseudo-Riemannian manifold need not be Einstein, it is known [7] that if \mathcal{M} is an indecomposable Jacobi–Videv manifold, then either ρ has only one real eigenvalue or ρ has two complex eigenvalues which are complex conjugates; such a manifold is said to be *pseudo-Einstein*. Clearly if ρ is nilpotent, then 0 is the only eigenvalue of ρ . This does not, however, imply \mathcal{M} is Jacobi–Videv as the following result shows:

Theorem 1.3. *Let $\{x, y, z, \bar{x}\}$ be coordinates on \mathbb{R}^4 . Let $\phi \in C^\infty(\mathbb{R})$. Assume $\phi''(y) \neq 0$ for all y . Let $\mathcal{M} := (\mathbb{R}^4, g)$ where $g(\partial_x, \partial_{\bar{x}}) = g(\partial_y, \partial_y) = g(\partial_z, \partial_z) = 1$ and $g(\partial_x, \partial_z) = 2\phi(y)$. Then:*

- (1) $\text{Rank}\{\rho\} = 3$, $\text{Rank}\{\rho^2\} = 2$, $\text{Rank}\{\rho^3\} = 1$, and $\rho^4 = 0$.
- (2) \mathcal{M} is not Jacobi–Videv.
- (3) $\alpha_\phi := \phi' \phi' \{\phi''\}^{-2}$ is a local isometry invariant of \mathcal{M} .
- (4) The following assertions are equivalent:
 - (a) \mathcal{M} is curvature homogeneous.
 - (b) \mathcal{M} is isometric to \mathcal{M}_b which is defined by $\phi(y) = e^{by}$ for $b > 0$.
 - (c) \mathcal{M} is homogeneous.

Let $\vec{x} = (x_1, x_2, x_3, x_4)$ be the canonical coordinates on \mathbb{R}^4 . One says that a pseudo-Riemannian manifold \mathcal{M} of signature $(2, 2)$ is a *Walker manifold* if it admits a parallel totally isotropic 2-plane field – see [1] for further details. Such a manifold is locally isometric to an example of the following form:

$$(1.a) \quad \begin{aligned} g(\partial_{x_1}, \partial_{x_3}) &= g(\partial_{x_2}, \partial_{x_4}) = 1, & g(\partial_{x_3}, \partial_{x_3}) &= g_{33}(\vec{x}), \\ g(\partial_{x_3}, \partial_{x_4}) &= g_{34}(\vec{x}), & g(\partial_{x_4}, \partial_{x_4}) &= g_{44}(\vec{x}). \end{aligned}$$

There are Jacobi–Videv manifolds of signature $(2, 2)$ which arise in the context of Walker geometry where we take $g_{33} = g_{44} = 0$. We refer to [16] for the proof of the following result and also for further information concerning Walker geometry:

Theorem 1.4. *Let $\mathcal{M} := (\mathbb{R}^4, g)$ be given by Eq. (1.a) where $g_{33} = g_{44} = 0$. Then \mathcal{M} is Jacobi–Videv if and only if $g_{34} = x_1 P(x_3, x_4) + x_2 Q(x_3, x_4) + S(x_3, x_4)$ where either*

- (1) $P/3 = Q/4$, i.e. $Qdx_3 + Pdx_4$ is a closed 1-form, or
- (2) There exist $(a, b, c) \neq (0, 0, 0)$ so $P = \frac{c}{a+bx_3+cx_4}$ and $Q = \frac{b}{a+bx_3+cx_4}$.

We remark that such a manifold is Einstein if and only if (2) holds. Thus Jacobi–Videv manifolds which are not Einstein may be created by taking (P, Q) to satisfy (1) but not (2); these will satisfy ρ is nilpotent but ρ need vanish identically.

We say that \mathcal{M} is skew–Videv if $\mathcal{R}(x, y)\rho = \rho\mathcal{R}(x, y)$ for all x, y . The following observation, which is of interest in its own right, will be crucial in our discussion:

Theorem 1.5. *Let \mathcal{M} be a pseudo-Riemannian manifold. The following assertions are equivalent:*

- (1) $R(\rho\xi_1, \xi_2, \xi_3, \xi_4) = R(\xi_1, \rho\xi_2, \xi_3, \xi_4) = R(\xi_1, \xi_2, \rho\xi_3, \xi_4) = R(\xi_1, \xi_2, \xi_3, \rho\xi_4)$ for all $\xi_i \in V$.
- (2) \mathcal{M} is skew–Videv.
- (3) \mathcal{M} is Jacobi–Videv.

We say \mathcal{M} is *Jacobi–Tsankov* if $\mathcal{J}(\xi_1)\mathcal{J}(\xi_2) = \mathcal{J}(\xi_2)\mathcal{J}(\xi_1)$ for all $\xi_1, \xi_2 \in V$ and that \mathcal{M} is *mixed–Tsankov* if $\mathcal{J}(\xi_1)\mathcal{R}(\xi_2, \xi_3) = \mathcal{R}(\xi_2, \xi_3)\mathcal{J}(\xi_1)$ for all $\xi_1, \xi_2, \xi_3 \in V$. As a scholium to the proof of Theorem 1.5, we will obtain the following

Theorem 1.6. *Let \mathcal{M} be a pseudo-Riemannian manifold. Then \mathcal{M} is Jacobi-Tsankov if and only if \mathcal{M} is mixed-Tsankov.*

The examples we have discussed in Theorems 1.2 and 1.4 involved nilpotent Ricci operators. We now discuss examples which are not Einstein, which are Jacobi-Videv, and where ρ is not nilpotent. To do this, it is convenient to pass to the algebraic setting. Let V be a finite dimensional vector space which is equipped with a non-degenerate inner product of signature (p, q) . Let $A \in \otimes^4 V^*$ be a 4-tensor. One says that $\mathfrak{M} := (V, \langle \cdot, \cdot \rangle, A)$ is a model if A has the symmetries of the Riemann curvature tensor:

$$\begin{aligned} A(v_1, v_2, v_3, v_4) &= A(v_3, v_4, v_1, v_2) = -A(v_2, v_1, v_3, v_4), \\ A(v_1, v_2, v_3, v_4) &+ A(v_2, v_3, v_1, v_4) + A(v_3, v_1, v_2, v_4) = 0. \end{aligned}$$

The associated curvature operator \mathcal{A} and bilinear Jacobi operator \mathcal{J} are characterized by the identities:

$$\begin{aligned} \langle \mathcal{A}(v_1, v_2)v_3, v_4 \rangle &= A(v_1, v_2, v_3, v_4), \\ \langle \mathcal{J}(v_1, v_2)v_3, v_4 \rangle &= \frac{1}{2}(A(v_3, v_1, v_2, v_4) + A(v_3, v_2, v_1, v_4)); \end{aligned}$$

the classical quadratic Jacobi operator being given by $\mathcal{J}(v) := \mathcal{J}(v, v)$.

Definition 1.7. Let $\mathfrak{M}_0 := (V_0, \langle \cdot, \cdot \rangle, A_0)$ be a Riemannian model. Let $\{e_i\}$ be an orthonormal basis for V_0 . Let $V_1 = V_0^+ \oplus V_0^-$ be two copies of V_0 with bases $\{e_i^+, e_i^-\}$. Let $\mathfrak{M}_1 := (V_1, \langle \cdot, \cdot \rangle, A_1)$ where

$$\begin{aligned} \langle e_i^+, e_i^+ \rangle &= 1, \quad \langle e_i^-, e_i^- \rangle = -1, \\ A_1(e_i^-, e_j^+, e_k^+, e_l^+) &= A_1(e_i^+, e_j^-, e_k^+, e_l^+) = A_1(e_i^+, e_j^+, e_k^-, e_l^+) \\ &= A_1(e_i^+, e_j^+, e_k^+, e_l^-) = A_0(e_i, e_j, e_k, e_l), \\ A_1(e_i^+, e_j^-, e_k^-, e_l^-) &= A_1(e_i^-, e_j^+, e_k^-, e_l^-) = A_1(e_i^-, e_j^-, e_k^+, e_l^-) \\ &= A_1(e_i^-, e_j^-, e_k^-, e_l^+) = -A_0(e_i, e_j, e_k, e_l). \end{aligned}$$

The following result may be used to construct examples of Jacobi-Videv models with $\rho^2 = -4s^2 \text{id}$:

Theorem 1.8. *Let \mathfrak{M}_0 be a Riemannian Einstein model with Einstein constant s and let \mathfrak{M}_1 be given by Definition 1.7. Then \mathfrak{M}_1 is a neutral signature model with $\rho_{\mathfrak{M}_1}^2 = -4s^2 \text{id}$ which is Jacobi-Videv.*

There are geometric examples of this phenomena. Again, we specialize the metric of Eq. (1.a) appropriately:

Theorem 1.9. *Let (x_1, x_2, x_3, x_4) be coordinates on \mathbb{R}^4 . Let $\mathcal{M} := (\mathbb{R}^4, g)$ where*

$$\begin{aligned} g(\partial_{x_1}, \partial_{x_3}) &= g(\partial_{x_2}, \partial_{x_4}) = 1, \quad g(\partial_{x_3}, \partial_{x_4}) = s(x_2^2 - x_1^2)/2, \\ g(\partial_{x_3}, \partial_{x_3}) &= sx_1x_2, \quad g(\partial_{x_4}, \partial_{x_4}) = -sx_1x_2. \end{aligned}$$

Then \mathcal{M} is locally symmetric (i.e. $\nabla R = 0$), \mathcal{M} is Jacobi-Videv, \mathcal{M} is skew-Videv, and $\rho^2 = -s^2 \text{id}$.

If $s = 1$, then $\rho^2 = -\text{id}$ so the Ricci operator ρ defines an almost complex structure on \mathcal{M} ; note that ρ is self-adjoint with respect to g and not skew-adjoint with respect to g and thus ρ is not unitary. Furthermore, since $\nabla R = 0$, $\nabla \rho = 0$. This manifold has been studied independently in a different context by E. García-Río [17]. They have informed us that both g and the associated Ricci tensor are irreducible $(2, 2)$ -metrics sharing the same Levi Civita connection and observed that this cannot happen either in the Riemannian nor the Lorentzian cases following results of [18].

Here is a brief outline to this paper. In Section 2, we establish Theorem 1.2, in Section 3, we establish Theorem 1.3, in Section 4 we establish Theorem 1.5, in Section 5 we establish Theorem 1.8, and in Section 6, we establish Theorem 1.9.

2. THE PROOF OF THEOREM 1.2

Let $\psi_{ij/a} := \partial_{y_a} \psi_{ij}$ and let $\psi_{ij/ab} := \partial_{y_a} \partial_{y_b} \psi_{ij}$. The non-zero Christoffel symbols of the first kind are given by:

$$\begin{aligned} g(\nabla_{\partial_{x_i}} \partial_{x_j}, \partial_{y_a}) &= \psi_{ij/a}, \quad \text{and} \\ g(\nabla_{\partial_{x_i}} \partial_{y_a}, \partial_{x_j}) &= g(\nabla_{\partial_{y_a}} \partial_{x_i}, \partial_{x_j}) = -\psi_{ij/a}. \end{aligned}$$

Let C^{ab} denote the inverse matrix. We adopt the Einstein convention and sum over repeated indices. The non-zero Christoffel symbols of the second kind are given by:

$$\begin{aligned} \nabla_{\partial_{x_i}} \partial_{x_j} &= C^{cd} \psi_{ij/c} \partial_{y_d}, \quad \text{and} \\ \nabla_{\partial_{x_i}} \partial_{y_a} &= \nabla_{\partial_{y_a}} \partial_{x_i} = -\delta^{kn} \psi_{ik/a} \partial_{\bar{x}_n}. \end{aligned}$$

Clearly $\mathcal{R}(\xi_1, \xi_2)\xi_3 = 0$ if any $\xi_i \in \text{Span}\{\partial_{\bar{x}_i}\}$. Furthermore,

$$\begin{aligned} \mathcal{R}(\partial_{x_i}, \partial_{x_j})\partial_{x_k} &= \nabla_{\partial_{x_i}} \nabla_{\partial_{x_j}} \partial_{x_k} - \nabla_{\partial_{x_j}} \nabla_{\partial_{x_i}} \partial_{x_k} \\ &= C^{cd} \delta^{rn} \{-\psi_{jk/c} \psi_{ir/d} + \psi_{ik/c} \psi_{jr/d}\} \partial_{\bar{x}_n}, \\ \mathcal{R}(\partial_{x_i}, \partial_{x_j})\partial_{y_a} &= \nabla_{\partial_{x_i}} \nabla_{\partial_{x_j}} \partial_{y_a} - \nabla_{\partial_{x_j}} \nabla_{\partial_{x_i}} \partial_{y_a} = 0, \\ \mathcal{R}(\partial_{x_i}, \partial_{y_a})\partial_{x_j} &= \nabla_{\partial_{x_i}} \nabla_{\partial_{y_a}} \partial_{x_j} - \nabla_{\partial_{y_a}} \nabla_{\partial_{x_i}} \partial_{x_j} = -C^{cd} \psi_{ij/ac} \partial_{y_d}, \\ \mathcal{R}(\partial_{x_i}, \partial_{y_a})\partial_{y_b} &= \nabla_{\partial_{x_i}} \nabla_{\partial_{y_a}} \partial_{y_b} - \nabla_{\partial_{y_a}} \nabla_{\partial_{x_i}} \partial_{y_b} = \delta^{kn} \psi_{ik/ab} \partial_{\bar{x}_n}, \\ \mathcal{R}(\partial_{y_a}, \partial_{y_b})\partial_{x_i} &= \nabla_{\partial_{y_a}} \nabla_{\partial_{y_b}} \partial_{x_i} - \nabla_{\partial_{y_b}} \nabla_{\partial_{y_a}} \partial_{x_i} \\ &= -\nabla_{\partial_{y_a}} \{\delta^{kn} \psi_{ik/b} \partial_{\bar{x}_n}\} + \nabla_{\partial_{y_b}} \{\delta^{kn} \psi_{ik/a} \partial_{\bar{x}_n}\} = 0, \\ \mathcal{R}(\partial_{y_a}, \partial_{y_b})\partial_{y_c} &= 0. \end{aligned}$$

The polarized Jacobi operator is given by

$$\mathcal{J}(\xi_1, \xi_2) : \xi_3 \rightarrow \frac{1}{2} \{\mathcal{R}(\xi_3, \xi_1)\xi_2 + \mathcal{R}(\xi_3, \xi_2)\xi_1\}.$$

The *Ricci form* $\rho(\xi_1, \xi_2) := \text{Tr}\{\mathcal{J}(\xi_1, \xi_2)\}$ vanishes if any $\xi_i \in \text{Span}\{\partial_{\bar{x}_i}\}$. Set

$$R_{ijk}{}^n = C^{cd} \delta^{rn} \{-\psi_{jk/c} \psi_{ir/d} + \psi_{ik/c} \psi_{jr/d}\}.$$

One has:

$$\begin{aligned} \mathcal{J}(\partial_{x_i}, \partial_{x_j})\partial_{x_k} &= \frac{1}{2}(R_{kij}{}^n + R_{kji}{}^n)\partial_{\bar{x}_n}, & \mathcal{J}(\partial_{x_i}, \partial_{x_j})\partial_{y_a} &= C^{cd} \psi_{ij/ac} \partial_{y_d}, \\ \mathcal{J}(\partial_{x_i}, \partial_{y_b})\partial_{x_j} &= -\frac{1}{2}C^{cd} \psi_{ij/bc} \partial_{y_d}, & \mathcal{J}(\partial_{x_i}, \partial_{y_a})\partial_{y_b} &= -\frac{1}{2}\delta^{kn} \psi_{ik/ab} \partial_{\bar{x}_n}, \\ \mathcal{J}(\partial_{y_a}, \partial_{y_b})\partial_{x_i} &= \delta^{kn} \psi_{ik/ab} \partial_{\bar{x}_n}, & \mathcal{J}(\partial_{y_a}, \partial_{y_b})\partial_{y_c} &= 0. \end{aligned}$$

Thus the non-zero components of the *Ricci form* $\rho(\xi_1, \xi_2) := \text{Tr}\{\mathcal{J}(\xi_1, \xi_2)\}$ may be seen to be:

$$\rho(\partial_{x_i}, \partial_{x_j}) = C^{ac} \psi_{ij/ac}.$$

Raising indices shows that the Ricci operator is given by:

$$(2.a) \quad \rho(\partial_{x_i}) = C^{ac} \delta^{jk} \psi_{ij/ac} \partial_{\bar{x}_k}, \quad \rho(\partial_{y_a}) = 0, \quad \rho(\partial_{\bar{x}_i}) = 0.$$

Since

$$\begin{aligned} \text{Range}\{\mathcal{J}(\xi_1, \xi_2)\} &\subset \text{Span}\{\partial_{y_a}, \partial_{\bar{x}_i}\} \subset \ker\{\rho\}, \\ \text{Range}\{\rho\} &\subset \text{Span}\{\partial_{\bar{x}_i}\} \subset \ker\{\mathcal{J}(\xi_1, \xi_2)\}, \end{aligned}$$

one has that $\rho\mathcal{J}(\xi_1, \xi_2) = \mathcal{J}(\xi_1, \xi_2)\rho = 0$. Thus \mathcal{M} is Jacobi–Videv. Since the Ricci operator is nilpotent, \mathcal{M} is pseudo-Einstein. Equation (2.a) shows \mathcal{M} is Einstein if and only if $\rho = 0$, i.e. if $C^{ab} \psi_{ij/ab} = 0$ for all ij . \square

3. THE PROOF OF THEOREM 1.3

Let (x, y, z, \bar{x}) be coordinates on \mathbb{R}^4 . Let $\phi = \phi(y)$ be a smooth function defined on a connected open subset of \mathbb{R} . We consider the metric

$$g(\partial_x, \partial_{\bar{x}}) = g(\partial_y, \partial_y) = g(\partial_z, \partial_z) = 1, \quad g(\partial_x, \partial_z) = 2\phi(y).$$

The Christoffel symbols of the first kind are given by:

$$\begin{aligned} g(\nabla_{\partial_x} \partial_z, \partial_y) &= g(\nabla_{\partial_z} \partial_x, \partial_y) = -\phi', \\ g(\nabla_{\partial_x} \partial_y, \partial_z) &= g(\nabla_{\partial_y} \partial_x, \partial_z) = g(\nabla_{\partial_y} \partial_z, \partial_x) = g(\nabla_{\partial_z} \partial_y, \partial_x) = \phi'. \end{aligned}$$

The non-zero covariant derivatives are therefore given by:

$$\begin{aligned} \nabla_{\partial_x} \partial_y &= \nabla_{\partial_y} \partial_x = \phi' \{ \partial_z - 2\phi \partial_{\bar{x}} \}, \\ \nabla_{\partial_y} \partial_z &= \nabla_{\partial_z} \partial_y = \phi' \partial_{\bar{x}}, \\ \nabla_{\partial_z} \partial_x &= \nabla_{\partial_x} \partial_z = -\phi' \partial_y. \end{aligned}$$

The action of the curvature operator may therefore be described by:

$$\begin{aligned} \mathcal{R}(\partial_x, \partial_y) \partial_x &= \nabla_{\partial_x} \phi' \{ \partial_z - 2\phi \partial_{\bar{x}} \} = -\phi' \phi' \partial_y, \\ \mathcal{R}(\partial_x, \partial_y) \partial_y &= -\nabla_{\partial_y} \phi' \{ \partial_z - 2\phi \partial_{\bar{x}} \} = -\phi'' \partial_z + \{ 2\phi'' \phi + \phi' \phi' \} \partial_{\bar{x}}, \\ \mathcal{R}(\partial_x, \partial_y) \partial_z &= \nabla_{\partial_x} \phi' \partial_{\bar{x}} + \nabla_{\partial_y} \phi' \partial_y = \phi'' \partial_y, \\ \mathcal{R}(\partial_x, \partial_z) \partial_x &= -\nabla_{\partial_x} \phi' \partial_y = -\phi' \phi' \{ \partial_z - 2\phi \partial_{\bar{x}} \}, \\ \mathcal{R}(\partial_x, \partial_z) \partial_y &= \nabla_{\partial_x} \phi' \partial_{\bar{x}} - \nabla_{\partial_z} \phi' \{ \partial_z - 2\phi \partial_{\bar{x}} \} = 0, \\ \mathcal{R}(\partial_x, \partial_z) \partial_z &= \nabla_{\partial_z} \phi' \partial_y = \phi' \phi' \partial_{\bar{x}}, \\ \mathcal{R}(\partial_y, \partial_z) \partial_x &= -\nabla_{\partial_y} \phi' \partial_y - \nabla_{\partial_z} \phi' \{ \partial_z - 2\phi \partial_{\bar{x}} \} = -\phi'' \partial_y, \\ \mathcal{R}(\partial_y, \partial_z) \partial_y &= \nabla_{\partial_y} \phi' \partial_{\bar{x}} = \phi'' \partial_{\bar{x}}, \\ \mathcal{R}(\partial_y, \partial_z) \partial_z &= -\nabla_{\partial_z} \phi' \partial_{\bar{x}} = 0. \end{aligned}$$

Consequently

$$\begin{aligned} \rho(\partial_x, \partial_z) &= -\phi'', \quad \rho(\partial_x, \partial_x) = 2\phi' \phi', \\ \rho : \partial_x &\rightarrow -\phi'' \partial_z + 2(\phi' \phi' + \phi \phi'') \partial_{\bar{x}}, \quad \rho : \partial_z \rightarrow -\phi'' \partial_{\bar{x}}. \end{aligned}$$

Since ρ is nilpotent, \mathcal{M} is pseudo-Einstein. We verify that \mathcal{M} is not Jacobi-Videv by computing:

$$\begin{aligned} \rho \mathcal{J}(\partial_x, \partial_y) \partial_y &= -\frac{1}{2} \rho \{ -\phi'' \partial_z + (2\phi'' \phi + \phi' \phi') \partial_{\bar{x}} \} = -\frac{1}{2} \phi'' \phi'' \partial_{\bar{x}}, \\ \mathcal{J}(\partial_x, \partial_y) \rho \partial_y &= 0. \end{aligned}$$

We now study the curvature tensor. We have

$$\begin{aligned} R(\partial_x, \partial_y, \partial_y, \partial_x) &= \phi' \phi', \quad R(\partial_x, \partial_z, \partial_z, \partial_x) = \phi' \phi', \quad R(\partial_y, \partial_z, \partial_z, \partial_y) = 0, \\ R(\partial_y, \partial_x, \partial_x, \partial_z) &= 0, \quad R(\partial_x, \partial_y, \partial_y, \partial_z) = -\phi'', \quad R(\partial_x, \partial_z, \partial_z, \partial_y) = 0, \end{aligned}$$

We say that a basis $\{X, Y, Z, \bar{X}\}$ is *normalized* if

$$\begin{aligned} (3.a) \quad g(X, \bar{X}) &= 1, \quad g(Y, Y) = 1, \quad g(Z, Z) = 1, \\ R(X, Y, Y, X) &= 0, \quad R(X, Z, Z, X) = \star, \quad R(Y, Z, Z, Y) = 0, \\ R(Y, X, X, Z) &= 0, \quad R(X, Y, Y, Z) = -1, \quad R(X, Z, Z, Y) = 0. \end{aligned}$$

Note that $R(X, Z, Z, X)$ is not specified. To create a normalized basis, we set

$$\begin{aligned} X &:= \varepsilon_1 \{ \partial_x + \delta_1 \partial_z - \frac{1}{2} (\delta_1^2 + 4\phi \delta_1) \partial_{\bar{x}} \}, \\ Y &:= \partial_y, \quad Z := \partial_z - (\delta_1 + 2\phi) \partial_{\bar{x}}, \quad \bar{X} := \varepsilon_1^{-1} \partial_{\bar{x}}. \end{aligned}$$

We then have:

$$\begin{aligned} g(X, \tilde{X}) &= g(Z, Z) = g(Y, Y) = 1, \\ R(X, Y, Y, X) &= \varepsilon_1^2 \{\phi' \phi' - 2\delta_1 \phi''\}, \\ R(Y, Z, Z, Y) &= R(Y, X, X, Z) = R(X, Z, Z, Y) = 0, \\ R(X, Y, Y, Z) &= -\varepsilon_1 \phi'', \quad R(X, Z, Z, X) = \varepsilon_1^2 \phi' \phi'. \end{aligned}$$

A normalized basis may then be defined by setting:

$$\varepsilon_1 := \{\phi''\}^{-1} \quad \text{and} \quad \delta_1 = \frac{1}{2} \phi' \phi' \{\phi''\}^{-1}.$$

We study the group of symmetries. We note that $\rho : X \rightarrow -Z$ and $\rho : Z \rightarrow -\tilde{X}$. Let $\{X_1, Y_1, Z_1, \bar{X}_1\}$ be another normalized model. Equation (3.a) yields

$$\begin{aligned} V_1 &:= \text{Span}_{\xi_i \in \mathbb{R}^4} \{\mathcal{R}(\xi_1, \xi_2) \xi_3\} = \text{Span}\{Y, Z, \bar{X}\} = \text{Span}\{Y_1, Z_1, \bar{X}_1\}, \\ V_1^\perp &= \text{Span}\{\bar{X}\} = \text{Span}\{\bar{X}_1\}, \\ \ker(\rho) &= \text{Span}\{Y, \bar{X}\} = \text{Span}\{Y_1, \bar{X}_1\}. \end{aligned}$$

Consequently, we may express:

$$\begin{aligned} X_1 &= a_1 X + a_2 Y + a_3 Z + a_4 \bar{X}, & Y_1 &= b_1 Y + b_2 \bar{X}, \\ Z_1 &= c_1 Y + c_2 Z + c_3 \bar{X}, & \bar{X}_1 &= d_1 \bar{X}. \end{aligned}$$

The conditions on the metric tensor yield $b_1 = \pm 1$, $c_1 = 0$, and $c_2 = \pm 1$. The condition $R(X_1, Y_1, Y_1, X_1) = 0$ shows $a_3 = 0$; as $R(Y_1, X_1, X_1, Z_1) = 0$, one also has $a_2 = 0$. Thus since $g(X_1, Y_1) = g(X_1, Z_1) = 0$ we also have $b_2 = c_3 = 0$. Since $g(X_1, X_1) = 0$, we also have $a_4 = 0$. The relations $g(X_1, \bar{X}_1) = 1$ and $R(X_1, Y_1, Y_1, Z_1) = -1$ then imply $a_1 = c_2$ and $d_1 = a_1^{-1}$. Thus

$$X_1 = a_1 X, \quad Y_1 = b_1 Y, \quad Z_1 = a_1 Z, \quad \bar{X}_1 = a_1^{-1} \bar{X} \quad \text{for} \quad a_1^2 = b_1^2 = 1.$$

The calculations performed above show that

$$\alpha_\phi := R(X, Z, Z, X) = \{\phi''\}^{-2} \phi' \phi'$$

is a local isometry invariant of \mathcal{M} ; Assertion (2) of Theorem 1.3 follows.

Assume that \mathcal{M} is curvature homogeneous. By Assertion (2), α_ϕ is constant. This implies that $\phi = ae^{by} + c$ where a , b , and c are suitably chosen real constants with $a \neq 0$ and $b \neq 0$ and consequently

$$ds^2 = dx \circ d\bar{x} + dy \circ dy + dz \circ dz + 2\{ae^{by} + c\}dx \circ dz.$$

We change variables setting $x = a^{-1}x_1$, $y = \text{sign}(b)y_1$, $z = z_1$, and $\bar{x} = a\bar{x}_1 - 2cz$. We show Assertion (3a) implies Assertion (3b) by checking:

$$\begin{aligned} ds^2 &= a^{-1}dx_1 \circ (ad\bar{x}_1 - 2cdz) + dy_1 \circ dy_1 + dz_1 \circ dz_1 \\ &\quad + 2(ae^{|b|y_1} + c)a^{-1}dx_1 \circ dz_1 \\ &= dx_1 \circ d\bar{x}_1 + dy_1 \circ dy_1 + dz_1 \circ dz_1 + 2e^{|b|y_1}dx_1 \circ dz_1. \end{aligned}$$

We therefore suppose $\phi(y) = e^{by}$ for $b > 0$. Given $(a_1, a_2, a_3, a_4) \in \mathbb{R}^4$, let $T(x, y, z, \bar{x}) = (e^{-2ba_2}x + a_1, y + a_2, z + a_3, e^{2ba_2}\bar{x} + a_4)$. Then

$$\begin{aligned} T^*ds^2 &= e^{-2ba_2}dx \circ e^{2ba_2}d\bar{x} + dy \circ dy + dz \circ dz \\ &\quad + 2e^{b(y+a_2)}e^{-2ba_2}dx \circ dz = ds^2. \end{aligned}$$

Consequently, T is an isometry of \mathcal{M} with $T(0, 0, 0, 0) = (a_1, a_2, a_3, a_4)$. This shows that the group of isometries acts transitively on \mathbb{R}^4 which completes the proof of Theorem 1.3. \square

4. THE PROOF OF THEOREM 1.5

We begin with a brief technical observation:

Lemma 4.1. *Let \mathfrak{M} be a model and let T be a self-adjoint linear map of V . The following conditions are equivalent:*

- (1) $A(T\xi_1, \xi_2, \xi_3, \xi_4) = A(\xi_1, T\xi_2, \xi_3, \xi_4) = A(\xi_1, \xi_2, T\xi_3, \xi_4) = A(\xi_1, \xi_2, \xi_3, T\xi_4)$
for all $\xi_i \in V$.
- (2) $T\mathcal{A}(\xi_1, \xi_2) = \mathcal{A}(\xi_1, \xi_2)T$ for all $\xi_i \in V$.
- (3) $T\mathcal{J}(\xi) = \mathcal{J}T(\xi)$ for all $\xi \in V$.

Proof. Let $\vec{\xi} := (\xi_1, \xi_2, \xi_3, \xi_4) \in V^4$. Let $\{i, j, k, l\}$ denote a permutation of the indices $\{1, 2, 3, 4\}$. Set $a_{ijkl} = a_{ijkl}(\vec{\xi}) := A(T\xi_i, \xi_j, \xi_k, \xi_l)$; a need not have the symmetries of an algebraic curvature tensor. Conditions (1), (2), and (3) are equivalent, respectively, to the following identities:

$$\begin{aligned} (4.a) \quad & a_{ijkl} = -a_{jikl} = a_{klij} = -a_{lkij}, \\ (4.b) \quad & a_{ijkl} = -a_{jikl}, \\ (4.c) \quad & a_{ijkl} + a_{ikjl} = a_{ljki} + a_{lkji}. \end{aligned}$$

Clearly Eq. (4.a) implies both Eqs. (4.b) and (4.c).

Conversely, suppose Eq. (4.b) holds. We may express:

$$\begin{aligned} a_{1234} &= \alpha_1, & a_{1342} &= \alpha_2, & a_{1423} &= -\alpha_1 - \alpha_2, \\ a_{2134} &= -\alpha_1, & a_{2341} &= \alpha_3, & a_{2413} &= \alpha_1 - \alpha_3, \\ a_{3124} &= \alpha_2, & a_{3241} &= -\alpha_3, & a_{3412} &= -\alpha_2 + \alpha_3, \\ a_{4123} &= \alpha_1 + \alpha_2, & a_{4231} &= \alpha_1 - \alpha_3, & a_{4312} &= \alpha_2 - \alpha_3. \end{aligned}$$

The Bianchi identity then implies $\alpha_3 = \alpha_1 + \alpha_2$ and Eq. (4.a) then follows.

Finally, suppose Eq. (4.c) holds. We use the Bianchi identities to express:

$$\begin{aligned} a_{1234} &= \alpha_1, & a_{1342} &= \alpha_2, & a_{1423} &= -\alpha_1 - \alpha_2, \\ a_{2134} &= -\beta_1, & a_{2341} &= \beta_1 + \beta_2, & a_{2413} &= -\beta_2, \\ a_{3124} &= \gamma_2, & a_{3241} &= -\gamma_1 - \gamma_2, & a_{3412} &= \gamma_1, \\ a_{4123} &= \delta_1 + \delta_2, & a_{4231} &= -\delta_2, & a_{4312} &= -\delta_1. \end{aligned}$$

We use Eq. (4.c) to derive the 6 identities:

$$\begin{aligned} \alpha_1 + 2\alpha_2 &= a_{1342} + a_{1432} = a_{2341} + a_{2431} = \beta_1 + 2\beta_2, \\ -2\alpha_1 - \alpha_2 &= a_{1243} + a_{1423} = a_{3241} + a_{3421} = -2\gamma_1 - \gamma_2, \\ \alpha_1 - \alpha_2 &= a_{1234} + a_{1324} = a_{4231} + a_{4321} = \delta_1 - \delta_2, \\ \beta_1 - \beta_2 &= a_{2143} + a_{2413} = a_{3142} + a_{3412} = \gamma_1 - \gamma_2, \\ -2\beta_1 - \beta_2 &= a_{2134} + a_{2314} = a_{4132} + a_{4312} = -2\delta_1 - \delta_2, \\ \gamma_1 + 2\gamma_2 &= a_{3124} + a_{3214} = a_{4123} + a_{4213} = \delta_1 + 2\delta_2. \end{aligned}$$

We set $\beta_i = \alpha_i + \varepsilon_i$, $\gamma_i = \alpha_i + \varrho_i$, and $\delta_i = \alpha_i + \sigma_i$. We then have:

$$\begin{aligned} 0 &= \varepsilon_1 + 2\varepsilon_2, & 0 &= -2\varrho_1 - \varrho_2, & 0 &= \sigma_1 - \sigma_2, \\ \varepsilon_1 - \varepsilon_2 &= \varrho_1 - \varrho_2, & -2\varepsilon_1 - \varepsilon_2 &= -2\sigma_1 - \sigma_2, & \varrho_1 + 2\varrho_2 &= \sigma_1 + 2\sigma_2. \end{aligned}$$

We use the first three equations to see $\varepsilon_1 = -2\varepsilon_2$, $\varrho_2 = -2\varrho_1$, and $\sigma_1 = \sigma_2$. The final 3 equations then become:

$$-3\varepsilon_2 = 3\varrho_1, \quad 3\varepsilon_2 = -3\sigma_1, \quad -3\varrho_1 = 3\sigma_1.$$

These equations imply $\varepsilon_i = \varrho_i = \sigma_i = 0$ and hence $\alpha_i = \beta_i = \gamma_i = \delta_i$ which completes the proof of the Lemma by showing Eq. (4.a) holds. \square

Theorem 1.5 now follows from Lemma 4.1 by taking $T = \rho$ and Theorem 1.6 follows by taking $T = \mathcal{J}(x)$.

5. THE PROOF OF THEOREM 1.8

We complexify and let $V_{\mathbb{C}} := V_0 \otimes_{\mathbb{R}} \mathbb{C}$. We extend (\cdot, \cdot) and A_0 to be complex multi-linear. Let $\{e_i\}$ be an orthonormal basis for V_0 . Let $\{e_i^+ := e_i, e_i^- := \sqrt{-1}e_i\}$ be a basis for the underlying real vector space $V_1 := V \oplus \sqrt{-1}V$. Let \Re and \Im denote the real and imaginary parts of a complex number, respectively. It is then immediate that

$$\langle \cdot, \cdot \rangle := \Re\{(\cdot, \cdot)\} \quad \text{and} \quad A_1(\cdot, \cdot, \cdot, \cdot) = \Im\{A_0(\cdot, \cdot, \cdot, \cdot)\}.$$

Consequently, A_1 has the appropriate curvature symmetries and defines an algebraic curvature tensor. We study the Ricci tensor by computing:

$$\begin{aligned} \rho_1(e_i^+, e_j^+) &= \sum_k \{A_1(e_i^+, e_k^+, e_k^+, e_j^+) - A_1(e_i^+, e_k^-, e_k^-, e_j^+)\} \\ &= \Im \sum_k \{A_0(e_i, e_k, e_k, e_j) + A_0(e_i, e_k, e_k, e_j)\} = 0, \\ \rho_1(e_i^-, e_j^-) &= \sum_k \{A_1(e_i^-, e_k^+, e_k^+, e_j^-) - A_1(e_i^-, e_k^-, e_k^-, e_j^-)\} \\ &= \Im \sum_k \{-A_0(e_i, e_k, e_k, e_j) - A_0(e_i, e_k, e_k, e_j)\} = 0, \\ \rho_1(e_i^+, e_j^-) &= \sum_k \{A_1(e_i^+, e_k^+, e_k^+, e_j^-) - A_1(e_i^+, e_k^-, e_k^-, e_j^-)\} \\ &= \Im \sum_k \{\sqrt{-1}A_0(e_i, e_k, e_k, e_j) + A_0(e_i, e_k, e_k, e_j)\} \\ &= 2 \sum_k A_0(v_i, v_k, v_k, v_j) = 2\rho_0(v_i, v_j) = 2s\delta_{ij} \end{aligned}$$

since \mathfrak{M}_0 is Einstein. We show $\rho_1^2 = -4s^2 \text{id}$ by computing:

$$\rho_1 : e_i^+ \rightarrow -2se_i^- \quad \text{and} \quad \rho_1 : e_i^- \rightarrow 2se_i^+.$$

We can view $\rho_1 = -2s\sqrt{-1}$ as a complex linear map of $V_{\mathbb{C}}$. Since we extended A_0 to be complex multi-linear, we compute

$$\begin{aligned} A_1(\rho_1 x, y, z, w) &= \Im\{A_0(\rho_1 x, y, z, w)\} = \Im\{A_0(-2s\sqrt{-1}x, y, z, w)\} \\ &= \Im\{-2s\sqrt{-1}A_0(x, y, z, w)\} = \Im\{A_0(x, y, z, -2s\sqrt{-1}w)\} \\ &= \Im\{A_0(x, y, z, \rho_1 w)\} = A_1(x, y, z, \rho_1 w). \end{aligned}$$

Theorem 1.5 now shows this model is Jacobi–Tsankov and Jacobi–Videv. □

6. THE PROOF OF THEOREM 1.9

Theorem 1.9 will follow from Theorem 1.5 and from the following result:

Lemma 6.1. *Let \mathcal{M} be the manifold of Theorem 1.9. Then*

- (1) \mathcal{M} is locally symmetric.
- (2) Let $\kappa = \frac{s}{2}$. Then $R_{1314} = \kappa$, $R_{1323} = -\kappa$, $R_{1424} = \kappa$, and $R_{2324} = -\kappa$.
- (3) $\rho\partial_{x_1} = -s\partial_{x_2}$, $\rho\partial_{x_2} = s\partial_{x_1}$, $\rho\partial_{x_3} = s\partial_{x_4}$, and $\rho\partial_{x_4} = -s\partial_{x_3}$.
- (4) $R(\rho\xi_1, \xi_2, \xi_3, \rho\xi_4) = -s^2 R(\xi_1, \xi_2, \xi_3, \xi_4)$ for all ξ_i .
- (5) $R(\rho\xi_1, \xi_2, \xi_3, \xi_4) = R(\xi_1, \xi_2, \xi_3, \rho\xi_4)$ for all ξ_i .

Proof. We used a Mathematica package developed by M. Brozos-Vázquez, J.C. Díaz-Ramos, E. García-Río and R. Vázquez-Lorenzo to establish Assertions (1)-(3). We establish Assertion (4) by computing:

$$\begin{aligned}
R_{1314} &= \kappa, & R_{\rho 1, \rho 3, 1, 4} &= -s^2 R_{2414} = -s^2 \kappa, & R_{\rho 1, 3, \rho 1, 4} &= s^2 R_{2324} = -s^2 \kappa \\
& & R_{\rho 1, 3, 1, \rho 4} &= s^2 R_{2313} = -s^2 \kappa, & R_{1, \rho 3, \rho 1, 4} &= -s^2 R_{1424} = -s^2 \kappa \\
& & R_{1, \rho 3, 1, \rho 4} &= -s^2 R_{1413} = -s^2 \kappa, & R_{1, 3, \rho 1, \rho 4} &= s^2 R_{1323} = -s^2 \kappa, \\
R_{1323} &= -\kappa, & R_{\rho 1, \rho 3, 2, 3} &= -s^2 R_{2423} = s^2 \kappa, & R_{\rho 1, 3, \rho 2, 3} &= -s^2 R_{2313} = s^2 \kappa, \\
& & R_{\rho 1, 3, 2, \rho 3} &= -s^2 R_{2324} = s^2 \kappa, & R_{1, \rho 3, \rho 2, 3} &= s^2 R_{1413} = s^2 \kappa, \\
& & R_{1, \rho 3, 2, \rho 3} &= s^2 R_{1424} = s^2 \kappa, & R_{1, 3, \rho 2, \rho 3} &= s^2 R_{1314} = s^2 \kappa, \\
R_{1424} &= \kappa, & R_{\rho 1, \rho 4, 2, 4} &= s^2 R_{2324} = -s^2 \kappa, & R_{\rho 1, 4, \rho 2, 4} &= -s^2 R_{2414} = -s^2 \kappa, \\
& & R_{\rho 1, 4, 2, \rho 4} &= s^2 R_{2423} = -s^2 \kappa, & R_{1, \rho 4, \rho 2, 4} &= -s^2 R_{1314} = -s^2 \kappa, \\
& & R_{1, \rho 4, 2, \rho 4} &= s^2 R_{1323} = -s^2 \kappa, & R_{1, 4, \rho 2, \rho 4} &= -s^2 R_{1413} = -s^2 \kappa, \\
R_{2324} &= -\kappa, & R_{\rho 2, \rho 3, 2, 4} &= s^2 R_{1424} = s^2 \kappa, & R_{\rho 2, 3, \rho 2, 4} &= s^2 R_{1314} = s^2 \kappa, \\
& & R_{\rho 2, 3, 2, \rho 4} &= -s^2 R_{1323} = s^2 \kappa, & R_{2, \rho 3, \rho 2, 4} &= s^2 R_{2414} = s^2 \kappa, \\
& & R_{2, \rho 3, 2, \rho 4} &= -s^2 R_{2423} = s^2 \kappa, & R_{2, 3, \rho 2, \rho 4} &= -s^2 R_{2313} = s^2 \kappa.
\end{aligned}$$

Since $\rho^2 = -s^2$, Assertion (5) follows from Assertion (4). \square

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